<u>**System of Equations**</u>

1. **(a)** (4, -2, 2) **(b)** (1, -1, 2) **(c)** (0, 1) **(d)** inconsistent **(e)** (-2-5t, 2+2t, t) , $t \in \mathbb{R}$.

2. **Method 1 (Determinant)**

$$\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & \beta & -1 \\ 6 & 7 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad 65 - 13\beta = 0 \quad \Rightarrow \quad \beta = 5$$

$$\Delta_x = 0 \Rightarrow \begin{vmatrix} \alpha & 1 & 3 \\ 1 & 5 & -1 \\ 2 & 7 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad -16 + 32\alpha = 0 \quad \Rightarrow \quad \alpha = 0.5$$

Also, $\Delta_y = \begin{vmatrix} 1 & 0.5 & 3 \\ 4 & 1 & -1 \\ 6 & 2 & 5 \end{vmatrix} = 0, \quad \Delta_z = \begin{vmatrix} 1 & 1 & 0.5 \\ 4 & 5 & 1 \\ 6 & 7 & 2 \end{vmatrix} = 0$

Since not all cofactors of $\Delta = 0$, the system of equations has infinite number of solutions.

Method 2 (Matrix)

Using augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & \alpha \\ 4 & \beta & -1 & 1 \\ 6 & 7 & 5 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & \alpha \\ 0 & \beta-4 & -13 & 1-4\alpha \\ 0 & 1 & -13 & 2-6\alpha \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & \alpha \\ 0 & 1 & -13 & 2-6\alpha \\ 0 & \beta-4 & -13 & 1-4\alpha \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & \alpha \\ 0 & 1 & -13 & 2-6\alpha \\ 0 & 0 & 13\beta-65 & 1-4\alpha-(2-6\alpha)(\beta-4) \end{array}\right)$$

In order that the system of equations has infinitely many solutions,

$$\begin{cases} 13\beta - 65 = 0 \\ 1 - 4\alpha - (2-6\alpha)(\beta-4) = 0 \end{cases} \Rightarrow \begin{cases} \beta = 5 \\ \alpha = 0.5 \end{cases}$$

3. $\begin{cases} (4-\lambda)x + 3y + z = 0 \\ 3x - (4+\lambda)y + 7z = 0 \\ x + 7y - (6+\lambda)z = 0 \end{cases}$ has nontrivial solutions $\Rightarrow$ $\Delta = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 3 & 1 \\ 3 & -4-\lambda & 7 \\ 1 & 7 & -6-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^3 + 6\lambda^2 - 75\lambda = 0 \quad \Rightarrow \quad \lambda(\lambda^2 + 6\lambda - 75) = 0 \quad \Rightarrow \lambda = 0, \quad -3 \pm 2\sqrt{21}$

Since λ is an integer , therefore $\lambda = 0$

The system is then reduced to $\begin{cases} 4x + 3y + z = 0 & \text{....(1)} \\ 3x - 4y + 7z = 0 & \text{....(2)} \\ x + 7y - 6z = 0 & \text{....(3)} \end{cases}$

Since (1) – (2) ≡ (3), eq. (3) is dependent on (1) and (2) and is redundant.

Solving (1) and (2), $\dfrac{x}{\begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix}} = \dfrac{y}{\begin{vmatrix} 1 & 4 \\ 7 & 3 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} 4 & 3 \\ 3 & -4 \end{vmatrix}} \Rightarrow \dfrac{x}{25} = \dfrac{y}{-25} = \dfrac{z}{-25} \Rightarrow x = -y = -z$

∴ The solution is (x, y, z) = (t, -t, -t) , where $t \in \mathbb{R}$.

4. **(a)** $\begin{vmatrix} p+1 & 1 & 1 \\ 1 & p+1 & 1 \\ 1 & 1 & p+1 \end{vmatrix} \overset{C_1 \to C_1 + C_2 + C_3}{=} \begin{vmatrix} p+3 & 1 & 1 \\ p+3 & p+1 & 1 \\ p+3 & 1 & p+1 \end{vmatrix} = (p+3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & p+1 & 1 \\ 1 & 1 & p+1 \end{vmatrix} = (p+3)\begin{vmatrix} 1 & 1 & 1 \\ 0 & p & 0 \\ 0 & 0 & p \end{vmatrix} = p^2(p+3)$

(b) **(i)** (E) has a unique solution

$$\Rightarrow \Delta = \begin{vmatrix} p+1 & 1 & 1 \\ 1 & p+1 & 1 \\ 1 & 1 & p+1 \end{vmatrix} = p^2(p+3) \neq 0 \Rightarrow p \neq 0 \text{ or } -3 \quad \Rightarrow \quad p \in \mathbb{R} \setminus \{0, -3\}$$

(ii) When $p = -3$, $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$.

Since not all cofactors of $\Delta = 0$, the system of equations has infinite number of solutions.

(E) becomes $\begin{cases} -2x + y + z = 0 & ....(1) \\ x - 2y + z = 0 & ....(2) \\ x + y - 2z = 0 & ....(3) \end{cases}$

(2) – (1), $3x - 3y = 0 \Rightarrow x = y$, (3) – (2), $3y - 3z = 0 \Rightarrow y = z$

$\therefore$ $(x, y, z) = (t, t, t)$, where $t \in \mathbb{R}$.

When $p = 0$, $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$.

the system of equations is reduced to only one equation $x + y + z = 0$

and it has infinite number of solutions: $(x, y, z) = (t_1, t_2, -t_1 - t_2)$, where $t_1, t_2 \in \mathbb{R}$.

(iii) no. For $p \in \mathbb{R} \setminus \{0, -3\}$, (E) has a unique solution.

For $p = \{0, -3\}$, (E) has infinite number of solutions

5. (E) $\begin{cases} (a-b-c)x + 2ay + 2a = 0 \\ 2bx + (b-c-a)y + 2b = 0 \\ 2cx + 2cy + (c-a-b) = 0 \end{cases} \Leftrightarrow$ (E') $\begin{cases} (a-b-c)x + 2ay + 2az = 0 \\ 2bx + (b-c-a)y + 2bz = 0 \\ 2cx + 2cy + (c-a-b)z = 0 \end{cases} \wedge \quad z = 1$

$\Rightarrow$ (E’) has non-trivial solution other than $(x, y, z) = (0, 0, 0)$

$$\Rightarrow \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = \begin{vmatrix} a+b+c & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3 = 0$$

$\Rightarrow$ $a + b + c = 0$ …. (*)

(1) If $a = b = c = 0$, (E) is reduced to $0x + 0y = 0$,

there are infinitely number of solutions : $(x, y) = (t_1, t_2)$, where $t_1, t_2 \in \mathbb{R}$.

(2) If not all $a, b, c = 0$, using (*), (E) is reduced to $x + y + 1 = 0$

there are infinitely number of solutions : $(x, y) = (t, -t - 1)$, where $t \in \mathbb{R}$.

6. **(a)** $A^2 - 4A + 11I = \begin{pmatrix} -7 & 8 \\ -16 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 \\ -4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

$A^2 = 4A - 11I$

$\Rightarrow$ $A^4 = (4A - 11I)^2 = 16A^2 - 88A + 121I = 16(4A - 11I) - 88A + 121I = -24A - 55I$

$$= -24\begin{pmatrix}1 & 2\\ -4 & 3\end{pmatrix} - 55\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-79 & -48\\ 96 & -127\end{pmatrix}$$

$$A^2 - 4A + 11\,I = 0 \quad \Rightarrow \quad A(A - 4\,I) = -11\,I \quad \Rightarrow A\left[\frac{1}{11}(4I - A)\right] = I$$

$$\Rightarrow A^{-1} = \frac{1}{11}(4I - A) = \frac{1}{11}\left[4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 2\\ -4 & 3\end{pmatrix}\right] = \frac{1}{11}\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$$

(b) $$\begin{cases} x + 2y - 3 = 0 \\ -4x + 3y + 5 = 0\end{cases} \Rightarrow \begin{pmatrix}1 & 2\\ -4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}3\\ -5\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 2\\ -4 & 3\end{pmatrix}^{-1}\begin{pmatrix}3\\ -5\end{pmatrix} = \frac{1}{11}\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}\begin{pmatrix}3\\ -5\end{pmatrix} = \frac{1}{11}\begin{pmatrix}19\\ 7\end{pmatrix}$$

7. **(a)** $$\begin{cases} x + y = 1 \\ (x \quad y)\,P = (x \quad y)\end{cases} \Leftrightarrow \begin{cases} y = 1 - x \\ (x \quad y)\begin{pmatrix}a & 1-a\\ 1-b & b\end{pmatrix} = (x \quad y)\end{cases} \Rightarrow (x \quad 1-x)\begin{pmatrix}a & 1-a\\ 1-b & b\end{pmatrix} = (x \quad 1-x)$$

$$\Rightarrow \big((a+b-2)x + (1-b) \quad -[(a+b-2)x + (1-b)]\big) = (0 \quad 0)$$

Since $a + b \neq 2$, $\exists$ a unique real 1×2 matrix $(x \quad y) = \left(\frac{b-1}{a+b-2}, \frac{a-1}{a+b-2}\right)$ satisfying the system.

(b) $$P - Q = \begin{pmatrix}a & 1-a\\ 1-b & b\end{pmatrix} - \begin{pmatrix}x & y\\ x & y\end{pmatrix} = \begin{pmatrix} a - \frac{b-1}{a+b-2} & 1 - a - \frac{a-1}{a+b-2} \\ 1 - b - \frac{b-1}{a+b-2} & b - \frac{a-1}{a+b-2}\end{pmatrix}$$

$$= \begin{pmatrix} \frac{a^2 + ab - 2a - b + 1}{a+b-2} & \frac{(1-a)(a+b-1)}{a+b-2} \\ \frac{(1-b)(a+b-1)}{a+b-2} & \frac{ab + b^2 - 2b + a - 1}{a+b-2}\end{pmatrix} = \begin{pmatrix} \frac{(a-1)(a+b-1)}{a+b-2} & \frac{(1-a)(a+b-1)}{a+b-2} \\ \frac{(1-b)(a+b-1)}{a+b-2} & \frac{(b-1)(a+b-1)}{a+b-2}\end{pmatrix}$$

$$= (a+b-1)\begin{pmatrix} \frac{a-1}{a+b-2} & \frac{1-a}{a+b-2} \\ \frac{1-b}{a+b-2} & \frac{b-1}{a+b-2}\end{pmatrix} = (a+b-1)\begin{pmatrix} 1 - \frac{b-1}{a+b-2} & -\frac{a-1}{a+b-2} \\ -\frac{1-b}{a+b-2} & 1 - \frac{b-1}{a+b-2}\end{pmatrix} = (a+b-1)\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}x & y\\ x & y\end{pmatrix}\right]$$

$\therefore \quad \lambda = a + b - 1$

$P^n - Q = \lambda^n(I - Q)$ can be proved by induction and noting :

1. $(x \quad y)\,P = (x \quad y) \Rightarrow \begin{pmatrix}x & y\\ x & y\end{pmatrix}P = \begin{pmatrix}x & y\\ x & y\end{pmatrix} \Rightarrow QP = Q$

2. $P^{n+1} - Q = P^{n+1} - QP = (P^n - Q)P = \lambda^n(I - Q)\,P = \lambda^n(P - QP) = \lambda^n(P - Q) = \lambda^n\,\lambda(I - Q) = \lambda^{n+1}\,(I - Q)$

8. **(a)** $$\Delta = \begin{vmatrix}0 & a & 1\\ 2 & 5 & 0\\ -2 & 1 & b\end{vmatrix} = 12 - 2ab = 2(6 - ab), \qquad ab \neq 6 \quad \Rightarrow \Delta \neq 0 \quad \Rightarrow \quad \text{inverse of A exists.}$$

$$\text{Min}(A) = \begin{pmatrix}5b & 2b & 12\\ ab-1 & 2 & 2a\\ -5 & -2 & -2a\end{pmatrix}, \ \text{Cof}(A) = \begin{pmatrix}5b & -2b & 12\\ 1-ab & 2 & -2a\\ -5 & 2 & -2a\end{pmatrix}, \quad \text{Adj}(A) = \begin{pmatrix}5b & 1-ab & -5\\ -2b & 2 & 2\\ 12 & -2a & -2a\end{pmatrix}$$

Inverse of A = $\dfrac{Adj(A)}{\Delta} = \dfrac{1}{12-2ab}\begin{pmatrix} 5b & 1-ab & -5 \\ -2b & 2 & 2 \\ 12 & -2a & -2a \end{pmatrix}$

(b) $\begin{cases} ay+z=2 \\ 2x+5y \quad =1 \\ -2x+y+bz=3 \end{cases} \Leftrightarrow \begin{pmatrix} 0 & a & 1 \\ 2 & 5 & 0 \\ -2 & 1 & b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & a & 1 \\ 2 & 5 & 0 \\ -2 & 1 & b \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{12-2ab}\begin{pmatrix} 5b & 1-ab & -5 \\ -2b & 2 & 2 \\ 12 & -2a & -2a \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{12-2ab}\begin{pmatrix} 10b-ab-14 \\ 8-4b \\ 24-8a \end{pmatrix}$$

$x = \dfrac{10-ab-14}{12-2ab}$, $y = \dfrac{4-2b}{6-ab}$, $z = \dfrac{12-4a}{6-ab}$, where $ab \neq 6$.

9. The augmented matrix : $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & -1 & -1 & 2 \\ 4 & 1 & 6 & c \end{array}\right) \underset{R_3 \to R_3 - 4R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & 0 \\ 0 & -3 & -2 & c-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & c-4 \end{array}\right)$

Suppose the system is consistent, $c - 4 = 0 \Rightarrow c = 4$.

When $c = 4$, we have $\begin{cases} x+y+2z=1 \\ -3y-2z=0 \end{cases} \Rightarrow$ (x, y, z) = (1 – 4t, –2t, 3t) , where $t \in \mathbb{R}$.

10. Let Δ_{ijk} be the determinant formed by the i, j, k equations of the given system. (i, j, k = 1, 2, 3, 4)

$$\Delta_{123} = \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix} = \begin{vmatrix} 1 & p & p^2 \\ 0 & q-p & q^2-p^2 \\ 0 & r-q & r^2-q^2 \end{vmatrix} = \begin{vmatrix} q-p & q^2-p^2 \\ r-q & r^2-q^2 \end{vmatrix} = (q-p)(r-q)\begin{vmatrix} 1 & q+p \\ 1 & r+q \end{vmatrix} = (p-q)(q-r)(r-p)$$

Replacing p = 1, respectively we get: $\Delta_{234} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix} = (1-q)(q-r)(r-1)$

Similarly, $\Delta_{134} = \begin{vmatrix} 1 & p & p^2 \\ 1 & 1 & 1 \\ 1 & r & r^2 \end{vmatrix} = (p-1)(1-r)(r-p)$, $\Delta_{124} = \begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & 1 & 1 \end{vmatrix} = (p-q)(q-1)(1-p)$

The necessary and sufficient conditions that the system has non-trivial solution(s) is

$$\Delta_{123} = \Delta_{234} = \Delta_{134} = \Delta_{124} = 0 \qquad \Leftrightarrow p = q = r = 1$$

11. **(a)** $\begin{vmatrix} \alpha & 1 & 0 \\ 1 & 0 & \beta \\ 0 & \beta & \alpha \end{vmatrix} = -\alpha - \alpha\beta^2 = -\alpha(\beta^2+1)$, $\alpha \neq 0$, $\beta \neq -1$.

(b) $AB = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & 0 & \beta \\ 0 & \beta & \alpha \end{pmatrix}\begin{pmatrix} -\beta^2 & -\alpha & \beta \\ -\alpha & \alpha^2 & -\alpha\beta \\ \beta & -\alpha\beta & -1 \end{pmatrix} = \begin{pmatrix} -\alpha\beta^2-\alpha & 0 & 0 \\ 0 & -\alpha\beta^2-\alpha & 0 \\ 0 & 0 & -\alpha\beta^2-\alpha \end{pmatrix} = -\alpha(\beta^2+1)\, I_3$

$$A^{-1} = \frac{1}{-\alpha(\beta^2+1)}B = \frac{1}{-\alpha(\beta^2+1)}\begin{pmatrix} -\beta^2 & -\alpha & \beta \\ -\alpha & \alpha^2 & -\alpha\beta \\ \beta & -\alpha\beta & -1 \end{pmatrix} = \frac{1}{\alpha(\beta^2+1)}\begin{pmatrix} \beta^2 & \alpha & -\beta \\ \alpha & -\alpha^2 & \alpha\beta \\ -\beta & \alpha\beta & 1 \end{pmatrix}$$

(c) If $\alpha \neq 0$, $\beta \neq -1$,

$$\begin{cases}\alpha x + y = 1\\ x + \beta z = 1\\ \beta y + \alpha z = 1\\ \alpha x + y - 2z = 1\end{cases} \Leftrightarrow \begin{cases}\begin{pmatrix}\alpha & 1 & 0\\ 1 & 0 & \beta\\ 0 & \beta & \alpha\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\\ \alpha x + y - 2z = 1\end{cases} \Leftrightarrow \begin{cases}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{1}{\alpha(\beta^2+1)}\begin{pmatrix}\beta^2 & \alpha & -\beta\\ \alpha & -\alpha^2 & \alpha\beta\\ -\beta & \alpha\beta & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = \frac{1}{\alpha(\beta^2+1)}\begin{pmatrix}\beta^2+\alpha-\beta\\ \alpha-\alpha^2+\alpha\beta\\ -\beta+\alpha\beta+1\end{pmatrix}\\ \alpha x + y - 2z = 1\end{cases}$$

By putting the solution of the first three eq. into the fourth, the system of equation has unique solution if

$$\alpha\frac{\beta^2+\alpha-\beta}{\alpha(\beta^2+1)} + \frac{\alpha-\alpha^2+\alpha\beta}{\alpha(\beta^2+1)} - 2\frac{-\beta+\alpha\beta+1}{\alpha(\beta^2+1)} = 1 \Rightarrow \alpha\beta + 1 = \beta$$

12. **(a)** $A = \begin{pmatrix}a+1 & 1 & 1\\ 1 & a+1 & 1\\ 1 & 1 & a+1\end{pmatrix}$, $|A| = 3a^2 + a^3 = a^2(a+3)$.

Min $A = \begin{pmatrix}2a+a^2 & a & -a\\ a & 2a+a^2 & a\\ -a & a & 2a+a^2\end{pmatrix}$, Cof $A = \begin{pmatrix}2a+a^2 & -a & -a\\ -a & 2a+a^2 & -a\\ -a & -a & 2a+a^2\end{pmatrix}$

Adj $A = \begin{pmatrix}2a+a^2 & -a & -a\\ -a & 2a+a^2 & -a\\ -a & -a & 2a+a^2\end{pmatrix}$

Inverse of A = $\frac{\text{Adj } A}{|A|} = \frac{1}{a(a+3)}\begin{pmatrix}a+2 & -1 & -1\\ -1 & a+2 & -1\\ -1 & -1 & a+2\end{pmatrix}$, where $a \neq 0, -3$.

(b) **(i)** $\begin{cases}(a+1)x + y + z = 1\\ x + (a+1)y + z = b\\ x + y + (a+1)z = b^2\end{cases}$ is equivalent to $\begin{pmatrix}a+1 & 1 & 1\\ 1 & a+1 & 1\\ 1 & 1 & a+1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ b\\ b^2\end{pmatrix}$

When $a \neq 0, -3$, inverse of A exists,

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}a+1 & 1 & 1\\ 1 & a+1 & 1\\ 1 & 1 & a+1\end{pmatrix}^{-1}\begin{pmatrix}1\\ b\\ b^2\end{pmatrix} = \frac{1}{a(a+3)}\begin{pmatrix}a+2 & -1 & -1\\ -1 & a+2 & -1\\ -1 & -1 & a+2\end{pmatrix}\begin{pmatrix}1\\ b\\ b^2\end{pmatrix} = \frac{1}{a(a+3)}\begin{pmatrix}a+2-b-b^2\\ -1+(a+2)b-b^2\\ -1-b+(a+2)b^2\end{pmatrix}$$

$$x = \frac{a+2-b-b^2}{a(a+3)}, \qquad y = \frac{-1+(a+2)b-b^2}{a(a+3)}, \quad z = \frac{-1-b+(a+2)b^2}{a(a+3)}$$

(ii) When $a = 0$, the system of equation becomes: $\begin{cases}x + y + z = 1\\ x + y + z = b\\ x + y + z = b^2\end{cases}$

When $b = 1$, the system becomes $x + y + z = 1$

and has infinite number of solution: $(x, y, z) = (1 - t_1 - t_2, t_1, t_2)$, where $t_1, t_2 \in \mathbb{R}$.

When $b \neq 1$, the system is inconsistent and has no solution.

(iii) When $a = -3$, the system of equation becomes: $\begin{cases}-2x + y + z = 1\\ x - 2y + z = b\\ x + y - 2z = b^2\end{cases}$

Adding the 3 equations, $b^2 + b + 1 = 0$.

Since $\Delta = 1^2 - 4(1)(1) < 0$. The quadratic equation has no real solution.

$\therefore$ The system of equations is inconsistent and has no solution.

13. **(a)** $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ p & q-p & r-p \\ p^2 & q^2-p^2 & r^2-p^2 \end{vmatrix} = (q-p)(r-p)\begin{vmatrix} 1 & 0 & 0 \\ p & 1 & 1 \\ p^2 & q+p & r+p \end{vmatrix}$

$= (q-p)(r-p)\begin{vmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ p^2 & q+p & r-p \end{vmatrix} = (q-p)(r-p)(r-q) = (p-q)(q-r)(r-p)$

(b) $\begin{pmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & q^2 & r^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ pq \\ pqr \end{pmatrix}$. By **(a)**, determinant of coeff. matrix $\neq 0$ since p, q and r are all distinct. Therefore there is unique solution. By Crammer's Rule,

$$x = \frac{\begin{vmatrix} p & 1 & 1 \\ pq & q & r \\ pqr & q^2 & r^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p\begin{vmatrix} 1 & 1 & 1 \\ q & q & r \\ qr & q^2 & r^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p\begin{vmatrix} 1 & 0 & 0 \\ q & 0 & r-q \\ qr & q(q-r) & r(r-q) \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{pq(q-r)}{(p-q)(r-p)}$$

$$y = \frac{\begin{vmatrix} 1 & p & 1 \\ p & pq & r \\ p^2 & pqr & r^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^2 & qr & r^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p\begin{vmatrix} 1 & 0 & 0 \\ p & q-p & r-p \\ p^2 & qr-p^2 & r^2-p^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p(r-p)\begin{vmatrix} 1 & 0 & 0 \\ p & q-p & 1 \\ p^2 & qr-p^2 & p+r \end{vmatrix}}{(p-q)(q-r)(r-p)}$$

$$= \frac{p(r-p)\left[(q-p)(p+r)-qr+p^2\right]}{(p-q)(q-r)(r-p)} = \frac{p(r-p)[pq-pr]}{(p-q)(q-r)(r-p)} = \frac{p^2}{p-q}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & p \\ p & q & pq \\ p^2 & q^2 & pqr \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p\begin{vmatrix} 1 & 0 & 0 \\ p & q-p & q-p \\ p^2 & q^2-p^2 & qr-p^2 \end{vmatrix}}{(p-q)(q-r)(r-p)} = \frac{p(q-p)\begin{vmatrix} 1 & 0 & 0 \\ p & 1 & q-p \\ p^2 & q+p & qr-p^2 \end{vmatrix}}{(p-q)(q-r)(r-p)}$$

$$= \frac{pq(p-q)(q-r)}{(p-q)(q-r)(r-p)} = \frac{pq}{r-p}$$

(c) **(i)** If $p = q \neq r$, using augmented matrix the system of equations becomes

$$\begin{pmatrix} 1 & 1 & 1 & p \\ p & p & r & p^2 \\ p^2 & p^2 & r^2 & p^2r \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & p \\ 0 & 0 & r-p & 0 \\ 0 & 0 & r^2-p^2 & p^2(r-p) \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r+p & p^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}$$

$\therefore$ The system is inconsistent.

(ii) If $p \neq q = r$, $\begin{pmatrix} 1 & 1 & 1 & p \\ p & q & q & pq \\ p^2 & q^2 & q^2 & pq^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & p \\ p-q & 0 & 0 & 0 \\ p^2-q^2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & p \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ $(x, y, z) = (0, p-t, t)$, $t \in \mathbb{R}$.

14. **(a)** $p \neq 0, 18$ **(b)** $q = -15$; $(x, y, z) = (t, 2t, 3t + \frac{5}{3})$, where $t \in \mathbb{R}$.

(c) $q = 15$; $(x, y, z) = (t, s, \frac{5-t-4s}{3})$, where $t, s \in \mathbb{R}$.